

## Opportunities for learning:

- ◆ This project promotes understanding of different voting systems by encouraging students to explore the mathematics on which they are based. It is hoped that this will empower them to use their own vote wisely in future life (or to make an informed decision about whether or not to vote).
- ◆ The project enables students to appreciate how particular voting systems favour a particular type of candidate and how the choice of voting system can influence the outcome of an election.
- ◆ This project encourages students to engage with the following mathematics subject content: recognising the properties and characteristics of arithmetic progressions, geometric progressions and other sequences, calculating the  $n$ th term, working systematically with combinations.

## General notes:

- ◆ Whilst this project does not shy away from real-life scenarios, it relates voting systems to students' own experiences, including choosing from a number of films on offer, in order to generate interest and to highlight how relevant and powerful mathematics can be.
- ◆ The ideas from the project might be used as a prelude to choosing a suitable method for electing class representatives or members of a school council.

## Notes and rich questions for 'Preference Voting' (Counting the Votes task 1):

- ◆ *Why can't you consider only 1st preferences in this case?*
- ◆ *Should more 1st preferences or fewer 5th preferences be considered more important?*
- ◆ *Should one 1st preference or two 2nd preferences be worth more?*

- ◆ You could organise a debate with some students arguing that the friends should choose Cleopatra, others arguing for Bobby, and others arguing for Tango. Encourage each group to justify their arguments using mathematics.
- ◆ Note that the five films on the task sheet are based on made-up titles. You might want to replace these titles with those of actual films.
- ◆ In small groups, you could ask students to nominate their own recent films and use preferences to decide which film they might want to go and see.
- ◆ *What questions do you need to consider in the design of your ballot papers?*
- ◆ *What would be the effect of nominating a film and its sequel, such as Rio and Rio 2?*
- ◆ *Hold your own classroom election for 'the best film of all time' and explore what happens when different voting systems are used to determine the winner. Which system was is fairest and why?*

## Notes and rich questions for 'Borda Points' (Counting the Votes task 2):

- ◆ *What do you notice about the sequence 1, 2, 3, 4, 5, 6, ... ? How does each term relate to the previous term? What is an arithmetic progression?*
- ◆ *What do you notice about the sequence 1, 2, 4, 8, 16, 32, ... ? How does each term relate to the previous term? What is a geometric progression?*
- ◆ *What would the  $n$ th term in each sequence be?*  
Note the  $n$ th term is the points that are awarded to each 1st preference if there are  $n$  candidates. For the arithmetic progression,  $n$ th term =  $n$ . For the geometric progression,  $n$ th term =  $2^{n-1}$ .

- ◆ *What formula connects the preference to the points awarded for each candidate (for  $n$  candidates)?* Note these formulae could be used to calculate the total points awarded to each candidate on a spreadsheet. For the arithmetic progression,  $\text{points} = n+1 - \text{preference}$ . For the geometric progression,  $\text{points} = 2^{n-\text{preference}}$
- ◆ The following four questions are ideal for prompting small group discussion.
- ◆ *What is meant by 'fair'? What type of winner will you be more likely to get using different systems?* Note that 'fair' is a highly subjective term and there is no method that can be considered to be truly 'fair'. People may have different views on which method is fairest, depending on what type of winner they prefer. Borda points with arithmetic weighting is more likely to result in a 'consensus' candidate winning than Borda points with geometric weighting. This should be emphasised through discussion.
- ◆ *What would be the effect of using a different arithmetic progression? Change the first term? Change the common difference?* Note this shouldn't make any difference to the results.
- ◆ *What would be the effect of using a different geometric progression? Change the first term? Change the common ratio?* Note this will make a difference - the greater the common ratio, the greater the weighting given to 1st preferences.
- ◆ *What other sequences could you use? What effect would they have? Which would be fairest?*
- ◆ Suggest other sequences such as square numbers, cube numbers, triangle numbers, prime numbers, Fibonacci sequence.

### Notes and rich questions for 'An Unusual Ballot' (Counting the Votes task 3):

- ◆ You could divide students up into groups and ask each group to start with a different voting system, before moving on to others. This will ensure that every method is explored by at least one group.
- ◆ Enlarging the ballot papers onto A3 paper will make it easier for students to work in groups.
- ◆ The methods are arranged roughly in increasing order of difficulty.
- ◆ **Relative majority** ('first past the post') - **Patil** is elected under this system (with 5 votes). Note in real life the ballot paper would not include preferences - electors would simply place an 'X' next to their first choice candidate.
- ◆ **Absolute majority** ('alternative vote') - **McAleese** is elected under this system. For this method (and Condorcet pair-wise counting), it is easier to distribute the ballot papers into piles (labelled with each candidate's name).
- ◆ Encourage students to think carefully about what to record. For the absolute majority method, it is essential to record how many votes each candidate has after each round in case there is a tie: Round 1 - Banda - 2 votes, McAleese - 3 votes, Patil - 5 votes, Roussef - 1 vote (Roussef is eliminated). Round 2 - Banda - 3 votes, McAleese - 3 votes, Patil - 5 votes (Banda is eliminated on 1st round votes). Round 3 - McAleese - 6 votes, Patil - 5 votes (McAleese has more than half of the votes)
- ◆ **Borda count with arithmetic weighting** - **Banda** is elected as follows: Banda - 30 points, McAleese - 25 points, Patil - 26 points, Roussef - 29 points.

◆ **Borda count with geometric weighting**

- **Patil** is elected as follows: Banda - 42 points, McAleese - 38 points, Patil - 46 points, Roussef - 39 points.

◆ **Condorcet pair-wise counting - Roussef**

is elected as she is preferred to every other candidate: Considering preferences for Banda and Roussef only: Banda - 5 votes, Roussef - 6 votes; Considering preferences for McAleese and Roussef only: McAleese - 5 votes, Roussef - 6 votes; Considering preferences for Patil and Roussef only: Patil - 5 votes, Roussef - 6 votes;

◆ Emphasise that there is not always a clear winner for Condorcet pair-wise counting and there must always be a back-up method in place in case this happens.

◆ For the 4 candidates, there will be 6 pairs of candidates to consider.

◆ *How many pairs of candidates would there be in an election with 3 candidates, 4 candidates, 5 candidates, 6 candidates, ... ,  $n$  candidates?* For  $n$  candidates, there would be  $\frac{1}{2}n(n-1)$  pairs - this is the  $(n-1)^{\text{th}}$  triangle number or  $1+2+3+\dots+(n-1)$

◆ Emphasise that this is an election for a single member to represent a group of people, e.g. it might be used to elect a Mayor, President, or General Secretary of a trade union.

◆ All of the methods are recognised preferential voting systems (see Electoral Reform Society website [www.electoral-reform.org.uk](http://www.electoral-reform.org.uk)) and are (or have been) used in real-life elections, e.g. the alternative vote is used in the Irish Presidential elections and the Borda count is used for the NBA's Most Valuable Player award. There are many more methods in use however these five have been chosen as they depend largely upon mathematical reasoning.

◆ Each of the four candidates is elected under at least one of the different voting methods. This suggests that the choice of

voting system can determine the outcome of the election.

◆ In a real-life election it is quite likely that the same candidate will be elected under different voting systems (hence the title of the task). However, this task has been designed carefully to highlight that this is not necessarily the case. The fact that the ballot papers are not 'typical' should be emphasised to students.

◆ *In light of the above: Who chooses the voting system? Who should choose the voting system?*

◆ A referendum in 2011 rejected a proposal to replace the 'first past the post' system with the 'alternative vote' system for counting votes in UK General Elections. A general apathy towards the referendum was blamed by some on a lack of public understanding of AV. See [www.electoral-reform.org.uk](http://www.electoral-reform.org.uk) for further explanation of these two systems and the pros and cons of each.

◆ There are many entertaining videos on the Internet promoting different voting systems, e.g. search for 'CGP Grey', 'AV cat'.

◆ Note that the names selected on the ballot papers are those of elected female Presidents (current or former) from Malawi, Ireland, India and Brazil.

**Possible extensions:**

◆ Design your own set of ballot papers so that different candidates win under different voting systems.

◆ Research and explore other voting systems not covered by these tasks, e.g. the supplementary vote (used in elections for London Mayor), and the two-round system (French Presidential elections).

◆ Design a spreadsheet that could be used to determine the winner of an election when the preferences from ballot papers are inputted.

- ◆ Compare the turnout in elections in the UK with those in other countries, e.g. South Africa, East Timor. *Is there any connection between turnout and how long a country has had democratic elections?*
- ◆ *Should voting be made compulsory (as it is in Australia)?*

### Case study:

- ◆ Rebecca's motivation for trying focusing on voting systems came from listening to sixth-formers' discussions that demonstrated a general antipathy towards elections. She perceived students' behaviour as one of the biggest barriers to bringing issues of social justice into the classroom and she abandoned her initial attempt at doing the 'An Unusual Ballot' task with her Year 9 class when they didn't respond well to being asked to work in a group. However, she began to appreciate this might be partly down to her own reluctance to 'let

go' and her tendency to be more structured with classes with more challenging behaviour.

She decided to give the activity a second go and was pleasantly surprised at how well students engaged with the activity and at how positive they were in their feedback about the opportunity the task provided for them to work together. Brian introduced the same task to a Year 8 group by showing students a humorous YouTube video, entitled 'Is your Cat confused about the alternative vote?' This video, produced for the 2011 referendum in the UK on the alternative vote, explained clearly the difference between the 'alternative vote' and 'first past the post' voting. Students demonstrated a critical appreciation of some of the issues involved in voting, e.g. one student came up himself with a question relating to who decides the voting system to be used and how important this decision was.

# PREFERENCE VOTING

## COUNTING THE VOTES 1

Imagine that three friends, Yaline, Brad and Alex, are going to the cinema together but they can't agree which film to see.

There are five films altogether to choose from.

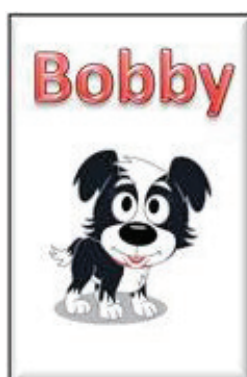
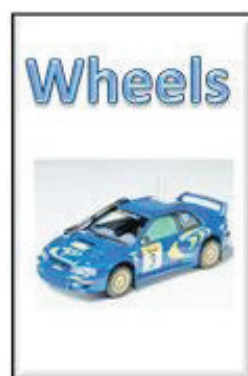
They decide to write down their preferences:

'1' for their first choice film, '2' for their second choice film, etc.

Each friend's preferences are shown below:

Yaline's preferences:

Film	Preference
Cleopatra	1
Wheels	5
Bobby	4
Tango	2
Cosmos	3



Brad's preferences:

Film	Preference
Cleopatra	5
Wheels	4
Bobby	3
Tango	2
Cosmos	1

Alex's preferences:

Film	Preference
Cleopatra	2
Wheels	4
Bobby	1
Tango	3
Cosmos	5



Group discussion:

- ◆ Which film should they see?
- ◆ Use only the information shown above.
- ◆ Justify your decision to others.



One way of deciding who wins in an election is to assign points for each preference.

Jean-Charles de Borda (1733-1799) was a French mathematician, physicist and political scientist who designed a preferential voting system that is still in use today.



Borda served as a ship's captain in the French navy and fought against the English in the American Revolutionary War in 1777-1778. As an engineer, he developed new instruments and methods used widely for navigation. He also constructed the standard metre that was used as the basis for the metric system of measurement.

After the French Revolution in 1789, there was an attempt to extend the use of the metric system to time and navigation. It was decided that a right angle should be divided into 100 degrees and a day should be divided into 10 hours, each hour being 100 minutes long. Borda was an enthusiast for the new system and constructed navigational instruments for use with the new units.

The new units of time didn't catch on and Napoleon abolished the Republican Calendar in 1806. However, the 'gradian' (one hundredth of a right angle) is still used today in engineering and the French military.

Borda had a ship, an island, a cape, an asteroid and a crater on the moon named after him. His name is one of 72 names inscribed on the Eiffel Tower.

### **Borda's preferential voting system:**

Voters in an election express preferences for each candidate, '1' for their first choice, '2' for their second choice, and so on.

A mathematical sequence is then used to award points for each preference.

The points awarded to each candidate are then added together.

The candidate with the largest total wins.

### **Borda count with arithmetic weighting:**

The sequence used is an arithmetic progression, for example 1, 2, 3, 4, 5, 6, ...

If there are 5 candidates, then 5 points are awarded for each 1st preference,

4 points for each 2nd preference, and so on down to 1 point for each 5th preference.

### **Borda count with geometric weighting:**

The sequence used is a geometric progression, for example 1, 2, 4, 8, 16, 32, ...

If there are 5 candidates, then 16 points are awarded for each 1st preference,


8 points for each 2nd preference, and so on down to 1 point for each 5th preference.

- ◆ Use Borda's preferential voting system to decide the winner in an election (you could use it to decide which film the friends should see in Counting the Votes 1).
- ◆ Discuss whether Borda count with arithmetic or geometric weighting is a fairer system to use for your chosen election. Justify your argument.
- ◆ Which other mathematical sequences could you use and how fair would they be?

These eleven ballot papers are from an imaginary election carried out using a preferential voting system:



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- ◆ Cut out and use the 11 ballot papers to decide which candidate would win the election under each method for counting the votes shown below.
- ◆ Discuss which method you think is fairest for counting the votes and why.

## Relative majority (or 'first past the post'):

Count only the 1<sup>st</sup> preferences and ignore all other preferences.  
The candidate with the most 1<sup>st</sup> preferences is elected.

## Absolute majority (or 'alternative vote'):

The candidate with the fewest 1<sup>st</sup> preferences is eliminated and their votes are redistributed according to the next available preference marked on the ballot papers. This process is repeated until one candidate has more than half of all votes. In the event of a tie, results of previous rounds should be taken into account.

## Borda count with arithmetic weighting:

See *Counting the Votes 2* for an explanation of this method.

## Borda count with geometric weighting:

See *Counting the Votes 2* for an explanation of this method.

## Condorcet Pair-Wise Counting:

Take each pair of candidates in turn and decide which of the two is preferred by considering only the preferences for those two candidates on the 11 ballot papers. If a candidate is preferred to all other candidates, then that candidate is elected.